Theoretical and Simulation Studies of CeC

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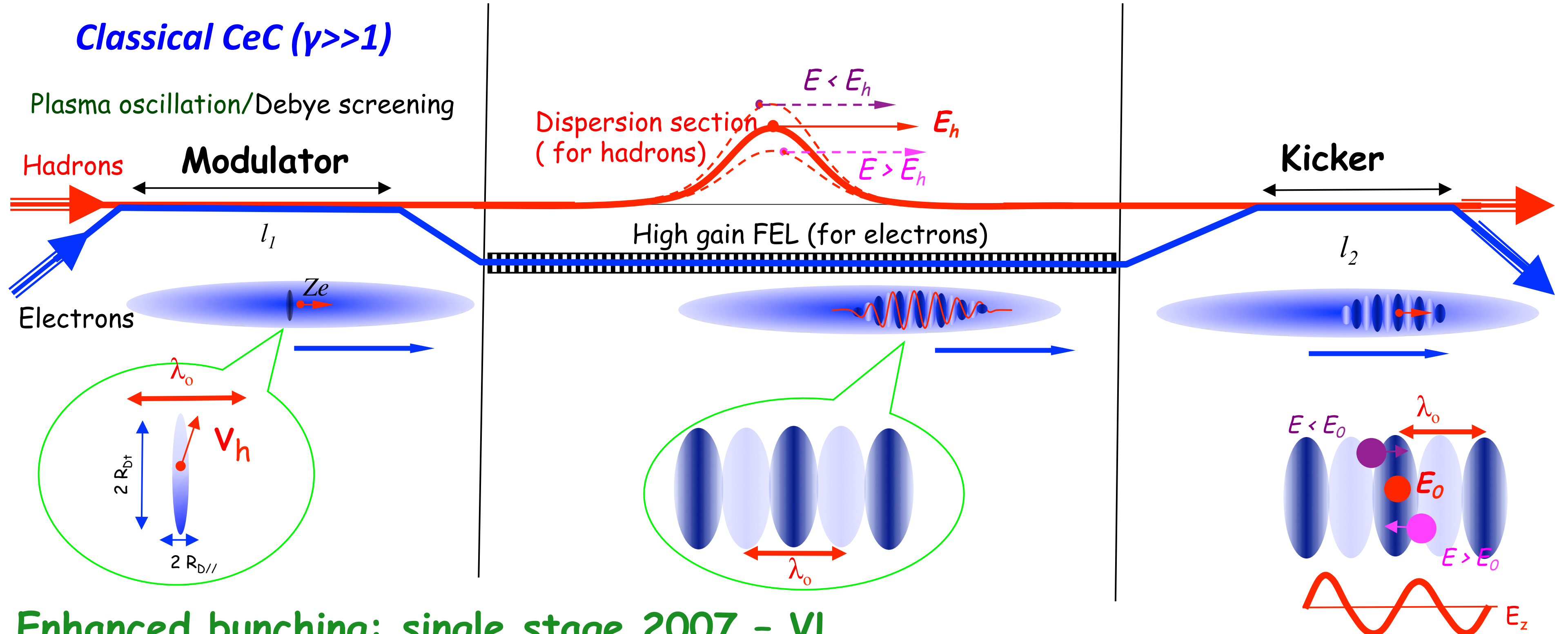
Outline

- Introduction
- Density and energy modulation
  - Results for cold electron beam
  - Results for warm electron beam
    - Analytical derivation of density modulation and comparison with simulation
    - Energy modulation and its influences
    - Influences of long-range longitudinal space charge field
- FEL amplification and its limitation
  - Amplification in an uniform beam
    - 1D FEL theory
    - Asymptotic behavior of 1D FEL amplifier in high-gain limit
    - FEL theory with non-collinear propogation of radiation
  - Longitudinal profile effects and beam conditioning
  - Saturation
- Analysis of kicker section
  - Analytical solution in an uniform electron beam
  - Reduction of field due to finite transverse modulation size
- Status of analytical studies
- Status of simulation
- Summary

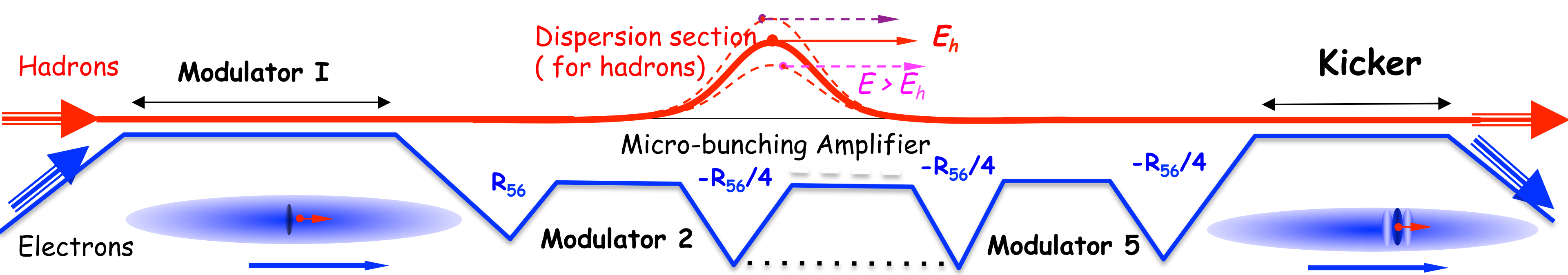
Introduction

Classical CeC (γ>>1)



Enhanced bunching: single stage 2007 – VL

Micro-bunching: Multi-stage 2013, D. Ratner, SLAC



Introduction Continued

- **Goals for analytical studies:**
  - Understanding the physical processes
  - Benchmarking simulation code
  - Estimation of cooling rate and its dependence on various parameters

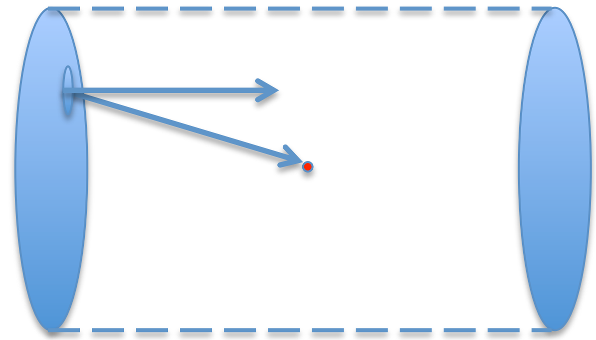
- **Goals for numerical simulation:**
  - Reproduces results for certain cases studied by analytical model to gain credibility (benchmarking)
  - Obtainning results for more general/realistic cases where analytical approach is prohibitely difficult
  - Provide hints for analytical modeling

Analytical Model for Modulator I: Cold Beam

- Density Modulation: $q = -Ze \cdot (1 - \cos\varphi_1)$ $\quad \varphi_1 = \omega_p l_1 / c\gamma_0$
- Energy Modulation ($\varphi_1 << 1$):

$$\frac{\delta E}{E}(z,r) = -Zr_e \frac{\gamma z}{\left(\gamma^2 z^2 + r^2\right)^{3/2}} \cdot c\Delta t$$

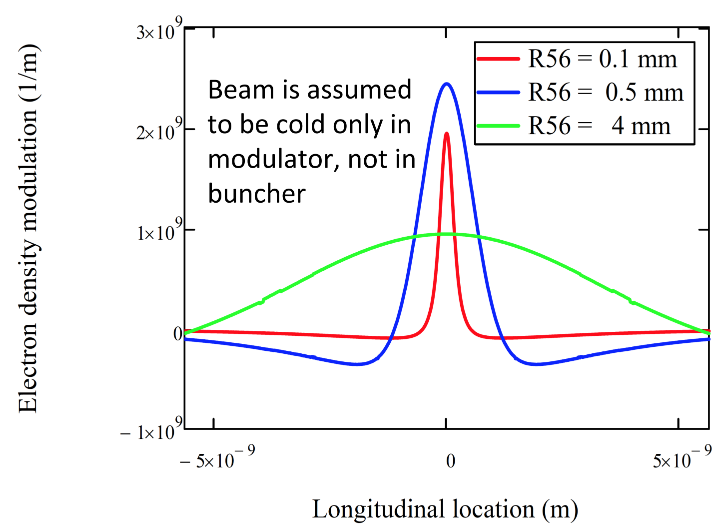
$$\left\langle \frac{\delta E}{E} \right\rangle \cong -2Z \frac{r_e}{a^2} \cdot \frac{L_{pol}}{\gamma} \cdot \left(\frac{z}{|z|} - \frac{z}{\sqrt{a^2/\gamma^2 + z^2}} \right)$$



- The energy spread has a simple form, which allows an analytical expression of the final charge density modulation for the buncher enhanced cooling scheme.

$$\tilde{\rho}(z) = 2\pi n_o \sigma_\gamma{}^2 D^2 \cdot \Psi_{u2}\left(\frac{\gamma_o z}{\sigma_\gamma |D| \sqrt{2}}, Z\frac{r_e L_{\mathrm{mod}}}{2\sqrt{2}\sigma_\gamma{}^3 D|D|}, \frac{a}{\sqrt{2}\sigma_\gamma |D|} \right)$$

$$\Psi_u(\zeta,\Xi,\tilde{a}) = \int_0^\infty q\,dq \left\{ \frac{Erf\left(q\left(1-\Xi q^{-3}\right)+\zeta\right) + Erf\left(q\left(1-\Xi q^{-3}\right)-\zeta\right)}{1-\Xi q^{-3}} \right.$$
$$\left. - \frac{Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)+\zeta\right) + Erf\left(q\left(1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}\right)-\zeta\right)}{1-\Xi\left(\sqrt{q^2+\tilde{a}^2}\right)^{-3}} \right\}$$



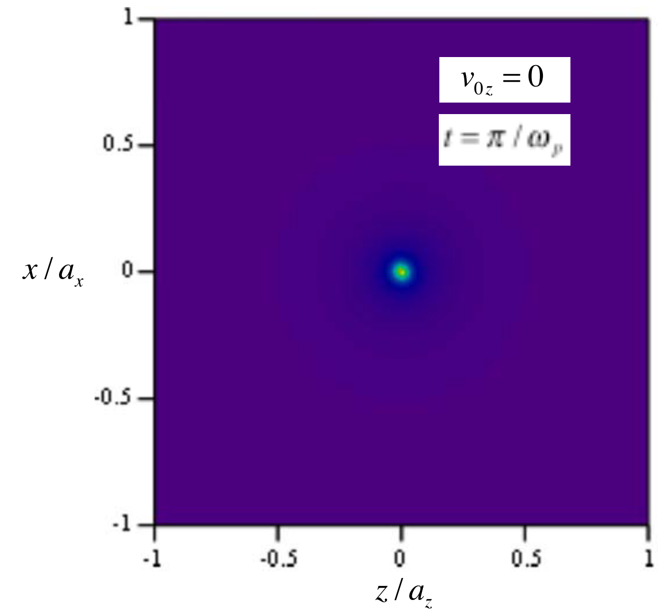
Analytical Model for Modulator II: Warm Beam

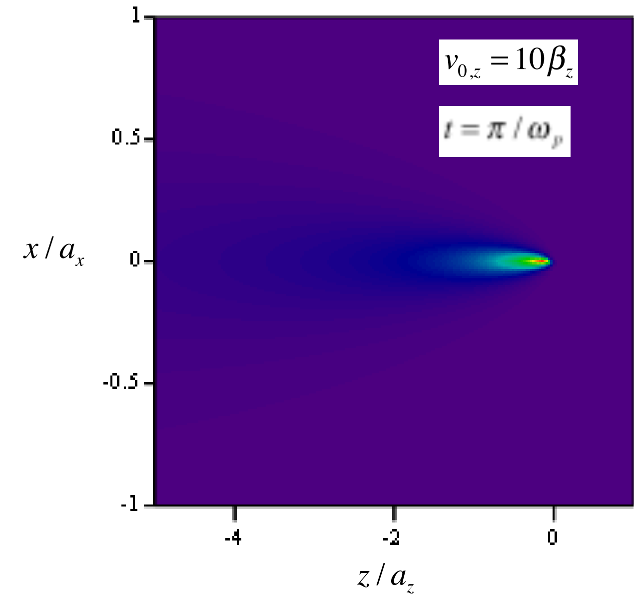
- The system is described by coupled linearized Vlasov-Poisson equation:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t} f_1(\vec{x},\vec{v},t)+\vec{v}\cdot\frac{\partial}{\partial \vec{x}} f_1(\vec{x},\vec{v},t)-\frac{e\vec{E}}{m_e}\frac{\partial}{\partial \vec{v}} f_0(\vec{v})=0\\ &\vec{\nabla}\cdot\vec{E}=\frac{1}{\varepsilon_o}\left\{Ze\delta(\vec{x})-e\tilde{n}_1(\vec{x},t)\right\}\end{aligned}}$$

$$f_0(\vec{v})=\frac{1}{\pi^2\beta_x\beta_y\beta_z}\left(1+\frac{v_x^2}{\beta_x^2}+\frac{v_y^2}{\beta_y^2}+\frac{v_z^2}{\beta_z^2}\right)^{-2}$$

$$\tilde{n}_1(\vec{x},t)=\frac{Z_i}{\pi^2 a_x a_y a_z}\int_0^{\omega_p t}\frac{\tau\sin\tau\cdot d\tau}{\left[\tau^2+\left(\frac{x}{a_x}+\frac{v_{0,x}}{\beta_x}\tau\right)^2+\left(\frac{y}{a_y}+\frac{v_{0,y}}{\beta_y}\tau\right)^2+\left(\frac{z}{a_z}+\frac{v_{0,z}}{\beta_z}\tau\right)^2\right]^2}$$

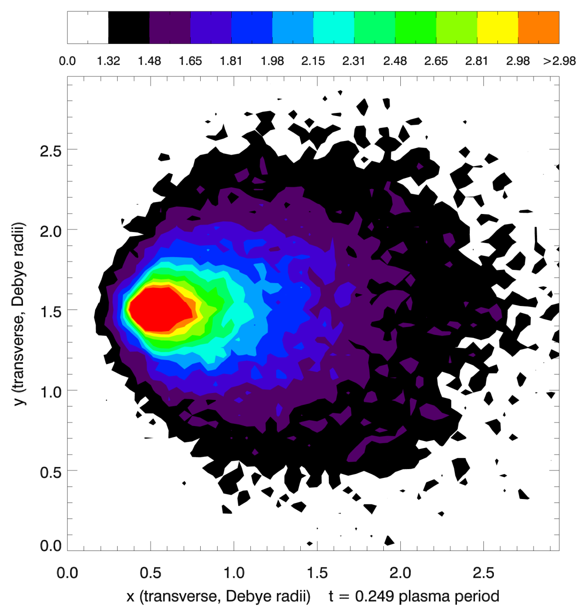


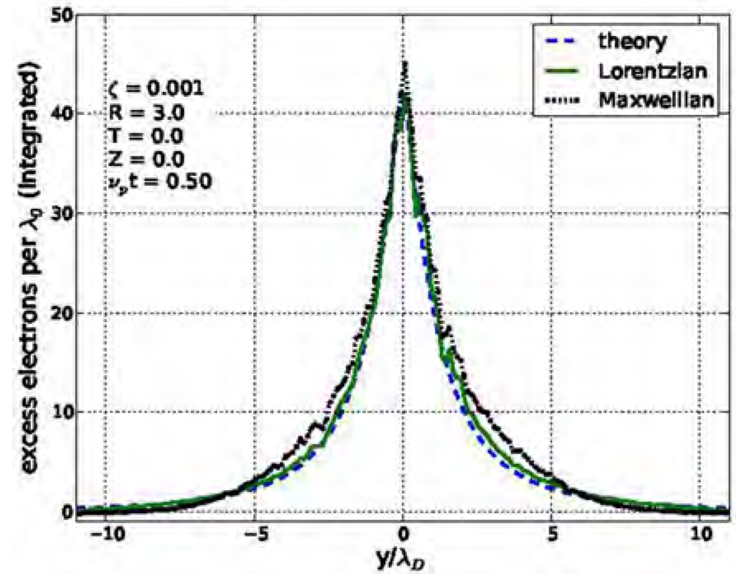


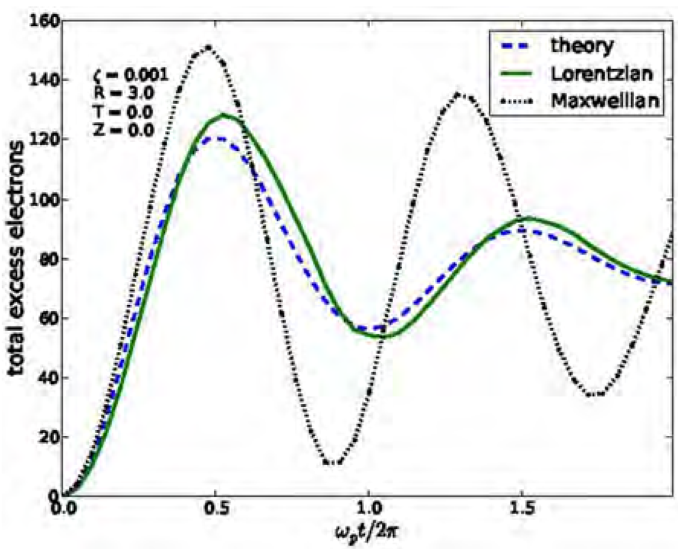
- The results is in the form of 1-D integral and hence is easy to evaluate.
- It has been used to benchmark multi-particle simulations.

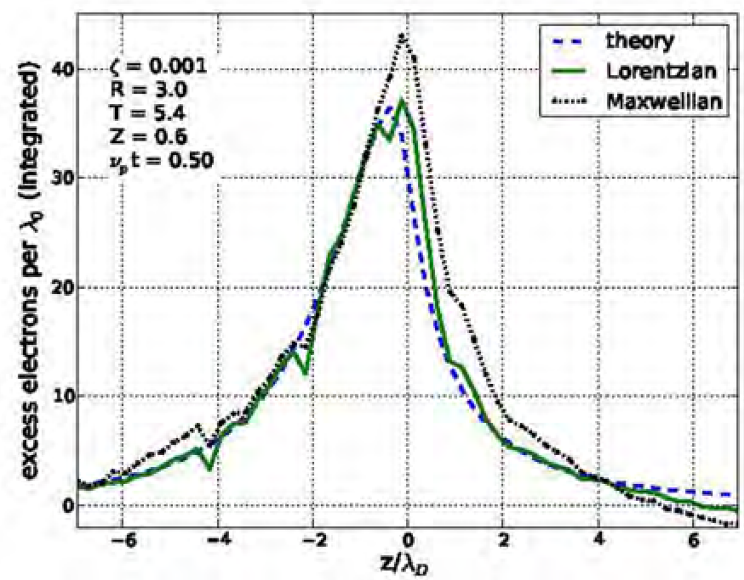
Simulations of modulator: uniform beam

- Simulation of the CeC modulator agree well with the analytical results.







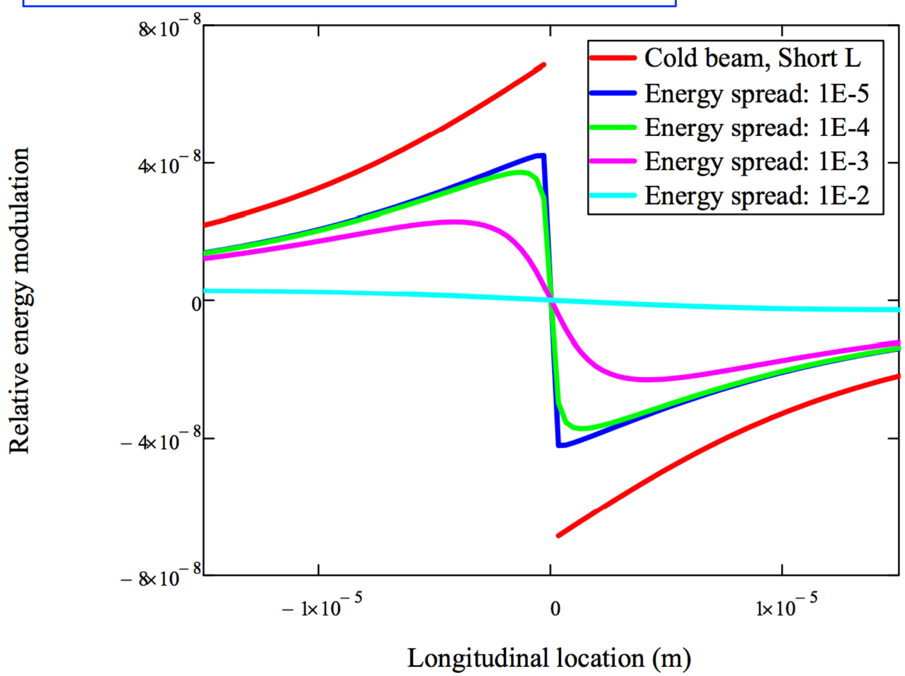


Analytical Model for Modulator II: Warm Beam continued

- Energy modulation: $\left\langle \frac{\delta E}{E_0} \right\rangle = \frac{\langle v_z \rangle}{c} = -\frac{1}{e n_0 \pi a^2 c} I_d\left(\gamma_0 z_l, \frac{L_{\text{mod}}}{\beta_0 \gamma_0 c}\right)$

$$I_d(z,t) = -\frac{Z_i e \omega_p^2}{\pi} \int_0^t d\tau \left(z + v_{0,z}\tau\right) \left\{ \frac{a_z \sin(\omega_p \tau)}{\left[\bar{\beta}^2\tau^2 + (z+v_{0,z}\tau)^2\right]\left[1 + \bar{\beta}^2\tau^2 + (z+v_{0,z}\tau)^2 / a^2\right]} - \cos(\omega_p \tau)\left[\frac{\arctan\left(\left|z+v_{0,z}\tau\right| / (\bar{\beta}\tau)\right)}{\left|z+v_{0,z}\tau\right|} - \frac{\arctan\left(\sqrt{(z+v_{0,z}\tau)^2 + a^2} / (\bar{\beta}\tau)\right)}{\sqrt{(z+v_{0,z}\tau)^2 + a^2}} \right] \right\}$$

Assuming effects from beam edge is small such that transversely current density does not deviate substantially from the infinite beam solution.



It reduces to the previously derived cold beam result at the corresponding limits:

$$\bar{\beta} = 0 \quad v_{0,z} = 0 \quad L_{\text{mod}} << \beta_0 \gamma_0 c / \omega_p$$

$$\left\langle \frac{\delta E}{E} \right\rangle \approx -2 Z_i \frac{r_e}{a^2} \frac{L_{\text{mod}}}{\gamma} \cdot \left[\frac{z_l}{|z_l|} - \frac{z_l}{\sqrt{z_l^{\,2} + a^2/\gamma^2}} \right].$$

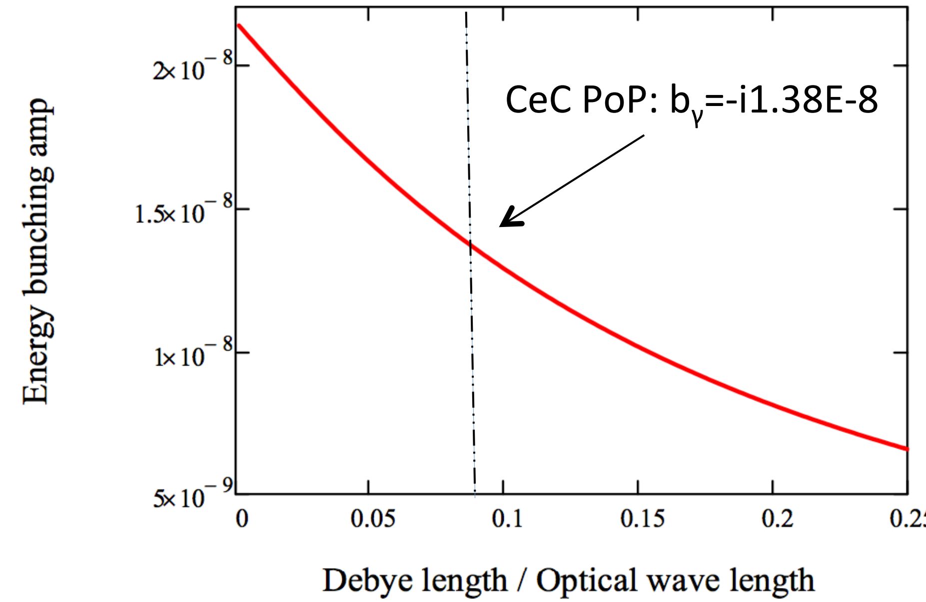
Influence of energy modulation on PoP experiment

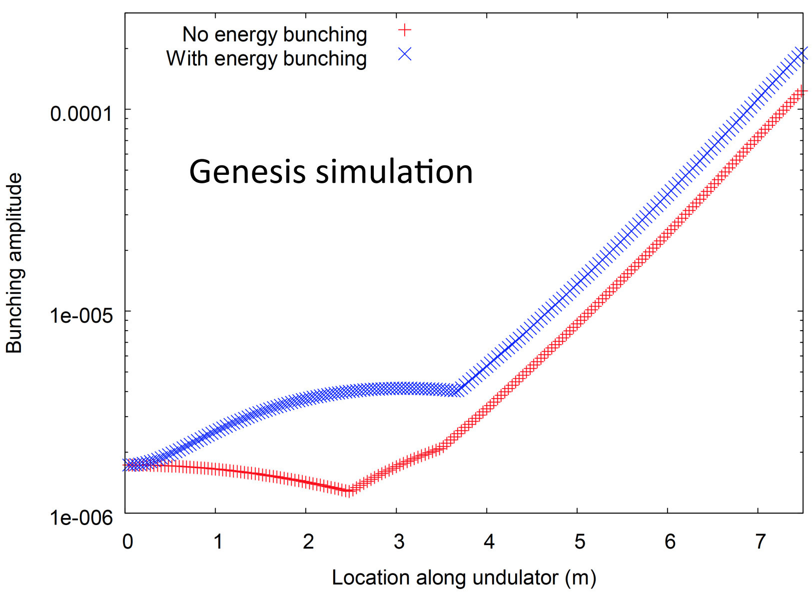
- The FEL simulation code, Genesis , is used to estimate the influences of energy modulation
- It is straightforward to calculated both the density and the energy bunching factor as input to Genesis simulation.

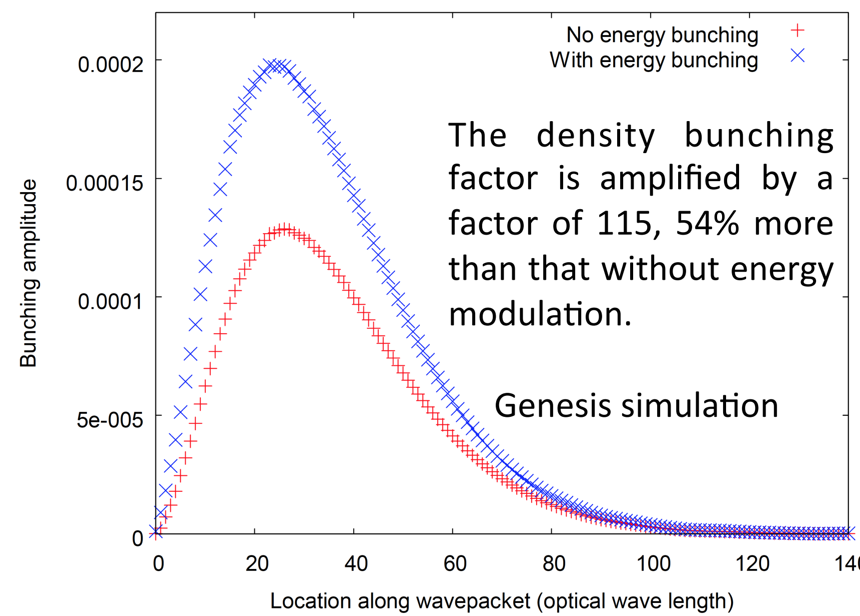
$$b = \frac{1}{N_\lambda}\frac{Z_i}{\pi}\int_{-\pi/\alpha}^{\pi/\alpha} d\bar{z}\exp\left(i\alpha\bar{z}\right)\int_0^{\omega_p t}\frac{\tau\sin\left(\tau\right)}{\left(\bar{z}+v_{0,z}\tau/\bar{\beta}\right)^2+\tau^2}d\tau;$$

$$\alpha \equiv 2\pi a_z/\left(\lambda_{opt}\gamma_0\right) \qquad \Rightarrow b = 1.73\times 10^{-6}$$

$$b_\gamma = \frac{1}{\lambda_{opt}}\int_{-\lambda_{opt}/2}^{\lambda_{opt}/2} e^{i2\pi z_l/\lambda_{opt}}\left\langle\frac{\delta E\left(z_l\right)}{E_0}\right\rangle dz_l$$

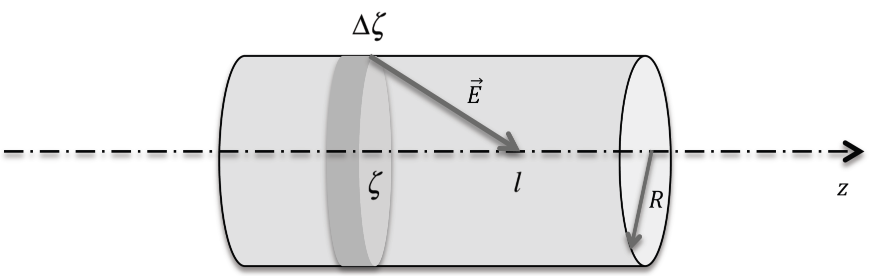






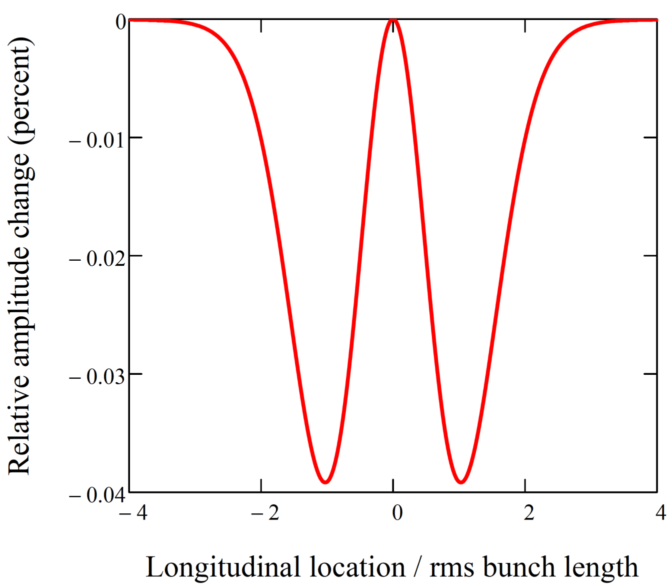
Effects from long-range longitudinal space charge

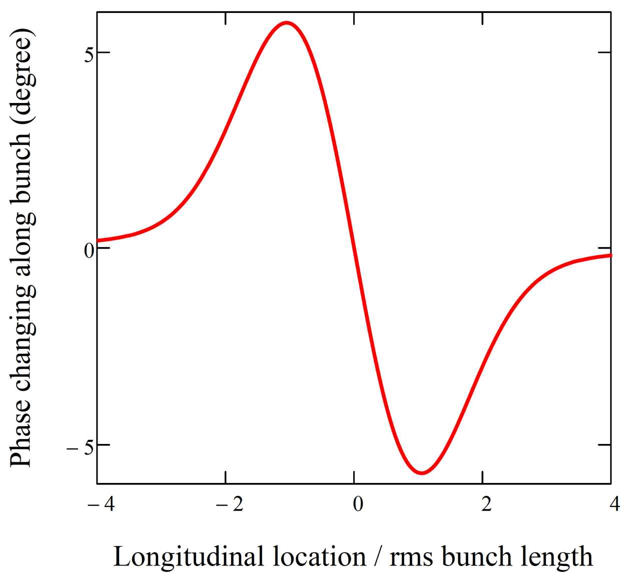
- For a bunched electron beam, the net space charge field does not vanish (except for the bunch center), which can affect the modulatio process.
- If the modulation happens locally, it is possible to treat the longitudinal space charge field as uniform and obtain the following result:

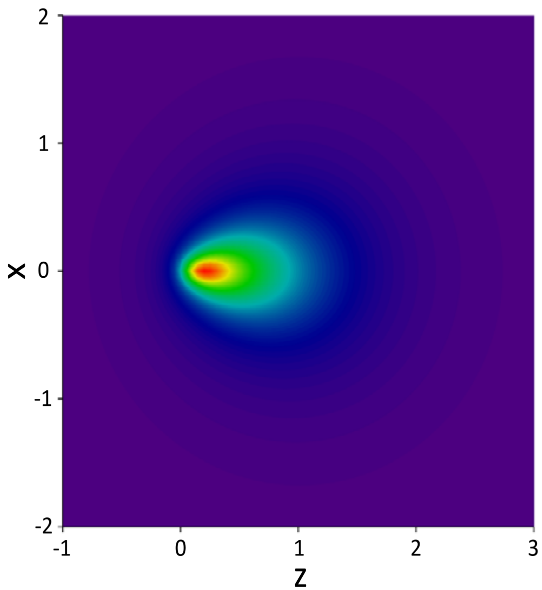


$$n_1(\vec{x},t)=\frac{Z_i}{\pi^2 r_x r_y r_z}\int_0^{\omega_p t}\psi\sin\psi\left[\psi^2+\left(\bar{x}-\bar{a}_x\psi\left(\omega_p t-\frac{\psi}{2}\right)+\bar{v}_{0,x}\psi\right)^2+\left(\bar{y}-\bar{a}_y\psi\left(\omega_p t-\frac{\psi}{2}\right)+\bar{v}_{0,y}\psi\right)^2+\left(\bar{z}-\bar{a}_z\psi\left(\omega_p t-\frac{\psi}{2}\right)+\bar{v}_{0,z}\psi\right)^2\right]^{-2}d\psi$$

- For CeC PoP, the influences of the longitudinal space charge field is negligible.

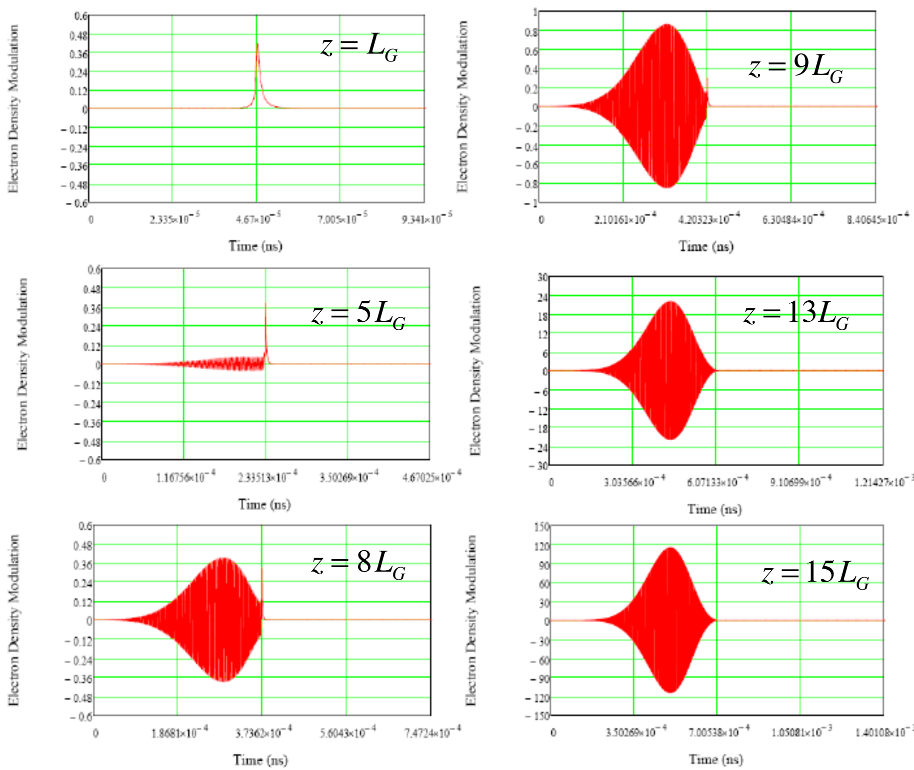






Analytical Model for FEL Amplifier I: 1D FEL

- The 1D FEL amplifier model with collinear radiation field and κ-1 (Lorentzian) energy distribution has been developed previously:



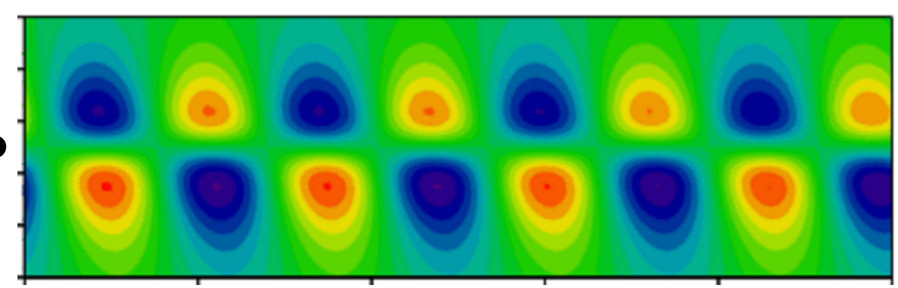
$$\tilde{j}_1(z) = -\left(\frac{\theta_s}{2\varepsilon_0 c}\right)^{-1}\left[A_1\lambda_1 e^{\lambda_1\hat{z}} + A_2\lambda_2 e^{\lambda_2\hat{z}} + A_3\lambda_3 e^{\lambda_3\hat{z}}\right]$$

$$\lambda^3 + 2\left(\hat{q} + i\hat{C}\right)\lambda^2 + \left[\hat{\Lambda}_p^2 + \left(\hat{q} + i\hat{C}\right)^2\right]\lambda - i = 0$$

$$F\left(\hat{P}\right) = \frac{1}{\pi\hat{q}}\frac{1}{1+\hat{P}^2/\hat{q}^2}$$

$$\tilde{f}_1\left(\hat{C},\hat{P},\hat{z}\right) = -\frac{e\theta_s n_0}{\pi\Gamma\varepsilon_0\rho}\frac{\hat{P}\hat{q}}{\left(\hat{q}^2+\hat{P}^2\right)^2}\sum_{i=1}^{3} A_i \frac{1+i\hat{\Lambda}_p^2\lambda_i}{\lambda_i + i\left(\hat{C}+\hat{P}\right)} e^{\lambda_i\hat{z}}$$

P

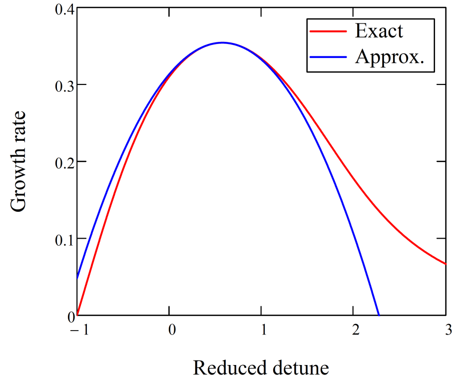


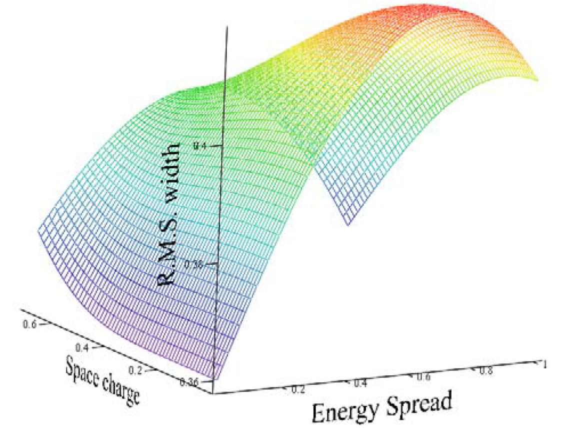
z

Asymptotic Behavior of 1-D FEL Amplifier at High Gain

- At high gain limit, the root of 1-D FEL dispersion relation can be expanded into

$$\lambda_1 = s_0 + s_1\left(\hat{C}-\hat{C}_0\right) + s_2\left(\hat{C}-\hat{C}_0\right)^2 + \dots$$





- Under this expansion, the current density modulation has Gaussian profile:

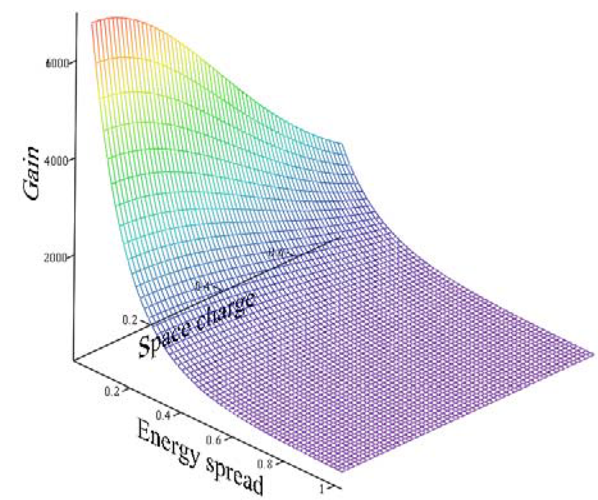
$$j_1(z,t) = \mathrm{Re}\left[\frac{Z_i e c k_0}{S\sqrt{\pi}} B_1\left(\hat{C}_0\right) s_0 \frac{e^{2 s_0 \rho k_w z}}{\sqrt{-s_2}} \sqrt{\frac{\rho}{2k_w z}}\, e^{i k_w z} e^{i k_0 (z-ct)} e^{-\frac{\left(t-t_p(z)\right)^2}{2\sigma_t^2}}\right]$$

with the width of $\sigma_{t,rms} = \dfrac{\left|s_2\right|}{\mathrm{Re}\left(s_2\right)\sigma_\omega}$

$$\sigma_\omega = 2c\gamma_z^2\sqrt{\frac{-\Gamma}{2\,\mathrm{Re}\left(s_2\right)z}} = \omega_0\sqrt{\frac{-\rho}{\mathrm{Re}\left(s_2\right)k_w z}}$$

and the maximal gain in the longitudinal electric field of

$$G = \frac{\left|E_{1,peak}\left(z,t_p\right)\right|}{E_0} = \frac{2\cdot 3^{\frac{1}{4}}}{\sqrt{\pi}}\rho\sqrt{\frac{L_g}{z}}\left|B_1\left(\hat{C}_0\right)s_0\right|\frac{e^{\frac{\mathrm{Re}(s_0)z}{\sqrt{3}L_g}}}{\sqrt{\left|-s_2\right|}}$$



Analytical Model for FEL Amplifier II: non-collinear Radiation

- We worked on the 1D FEL amplifier model with non-collinear radiation field and κ-2 energy distribution. This model includes diffraction effects, which becomes important when the transverse Debye radius is much smaller than the beam size.).

$$\nabla_\perp^2 \tilde{E}(z,r_\perp,C) + 2i\frac{\omega}{c}\frac{\partial}{\partial z}\tilde{E}(z,r_\perp,C) = ij_0(r_\perp)\int_0^z dz'\left\{\frac{2\pi e}{c^2}\theta_s^2\omega\tilde{E}(z',r_\perp,C) + \frac{4\pi e}{\omega}\left[\nabla_\perp^2\tilde{E}(z',r_\perp,C) + 2i\frac{\omega}{c}\frac{\partial}{\partial z}\tilde{E}(z',r_\perp,C)\right]\right\}$$

$$\times\int_{-\infty}^{\infty} dP\frac{dF}{dP}\exp\left[i\left(C + \frac{\omega}{\gamma_z^2\varepsilon_0 c}\right)(z'-z)\right]$$

$$j_0(r_\perp) = j_0$$

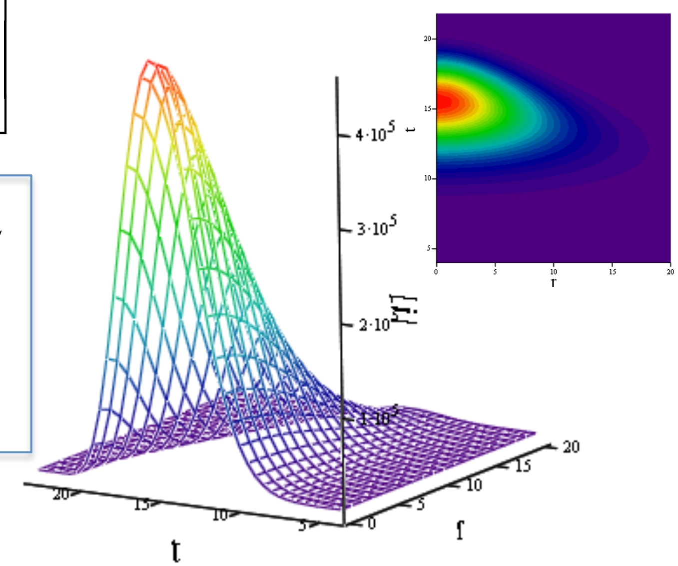
$$F(\hat{P}) = \frac{2}{\pi\hat{q}}\frac{1}{\left(1+\hat{P}^2/\hat{q}^2\right)^2}$$

$$\tilde{f}_1(z,t,x,y,P) = \frac{ec\delta(P)}{2\pi\sigma_\perp^2\sqrt{2\pi}\sigma_t}e^{-\frac{r^2}{2\sigma_\perp^2}}e^{-\frac{(z-\beta ct)^2}{2\sigma_z^2}}$$

$$\hat{q} = 0.1 \quad \hat{\sigma}_\perp = 2.32$$

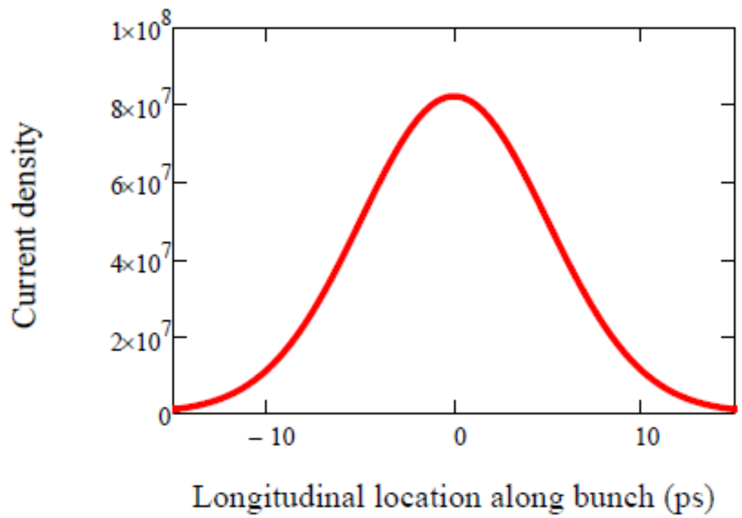
$$\hat{\Lambda}_p = 0 \quad \hat{z} = 21$$

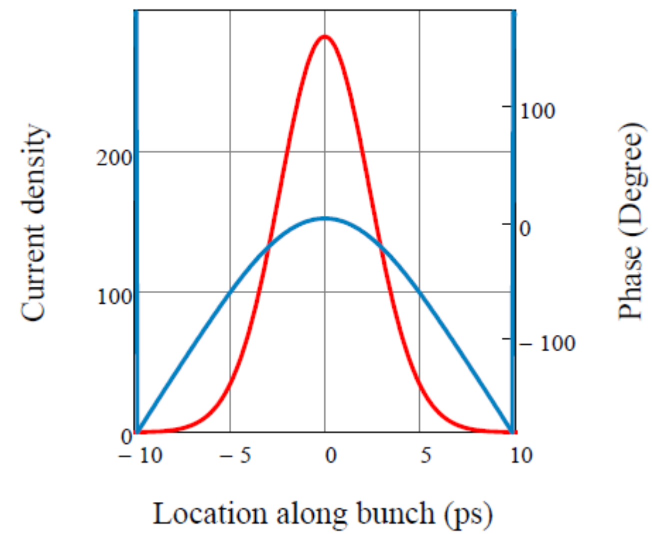
$$\hat{\sigma}_t = 1.37\times 10^{-5}$$

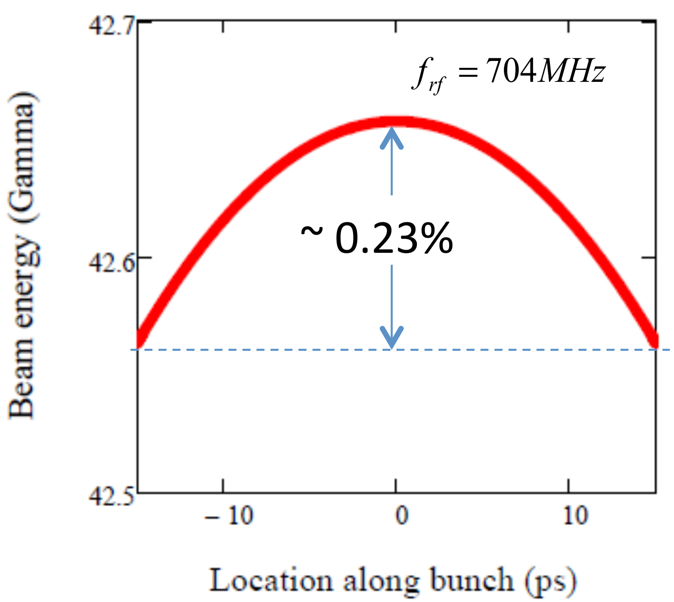


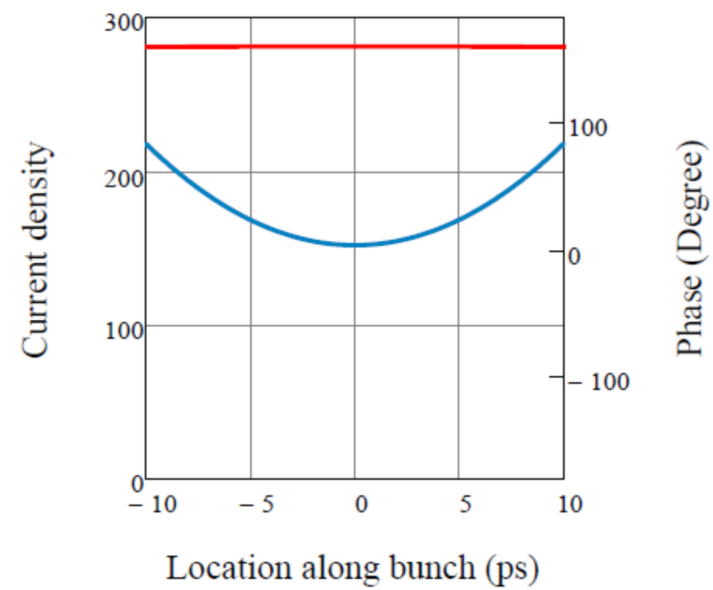
$$\tilde{j}_1(\vec{x},t) = -\frac{ec\Gamma^3\gamma_z^4}{2\pi^2\rho}e^{-i\hat{k}_w\xi}\sum_{(i,j,k,l)}\int_{-\infty}^{\infty} d\hat{C}_{3d}e^{-i2\gamma_z^2(\hat{z}-c\hat{t})\hat{C}_{3d}}I_x(\hat{r},\hat{C}_{3d})\frac{\lambda_i\left(B_{jkl} + \frac{\hat{q}^3}{\lambda_i + i\hat{C}_{3d}}\right)e^{\lambda_i\hat{z}} + \frac{i\hat{q}^3\hat{C}_{3d}}{\lambda_i + i\hat{C}_{3d}}e^{-i\hat{C}_{3d}\hat{z}}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}$$

One Application of the FEL Theory in a Uniform Beam:
Longitudinal Profile Effects and Beam Conditioning



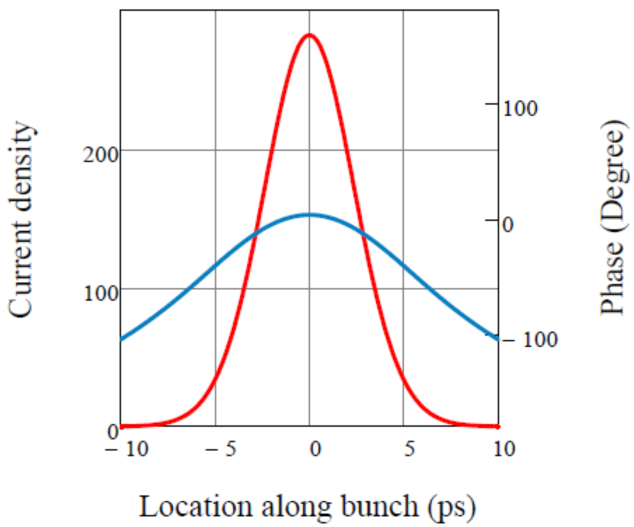




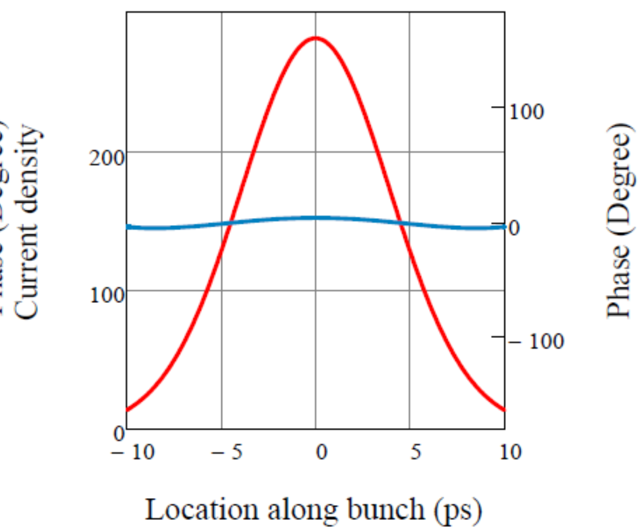


Beam Conditioning:
Minimizing Phase Variation by Varing Bunch Length

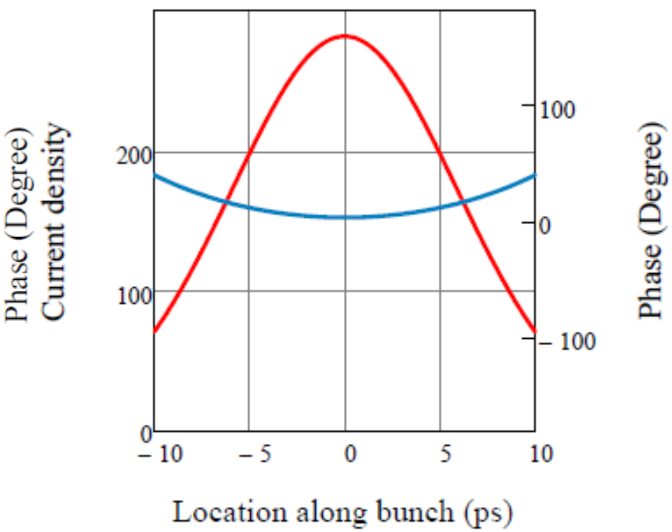
5 ps rms bunch length



8 ps rms bunch length



12 ps rms bunch length



- With the same peak current density, there is an optimum bunch length to minimize phase variation of the wave packet with respect to ions located at different portion of the electron bunch.
- For the specific parameters that we consider, the optimum bunch length is around 8 ps, which reduce the phase variation to the level of 5 degree .

Genesis Simulation

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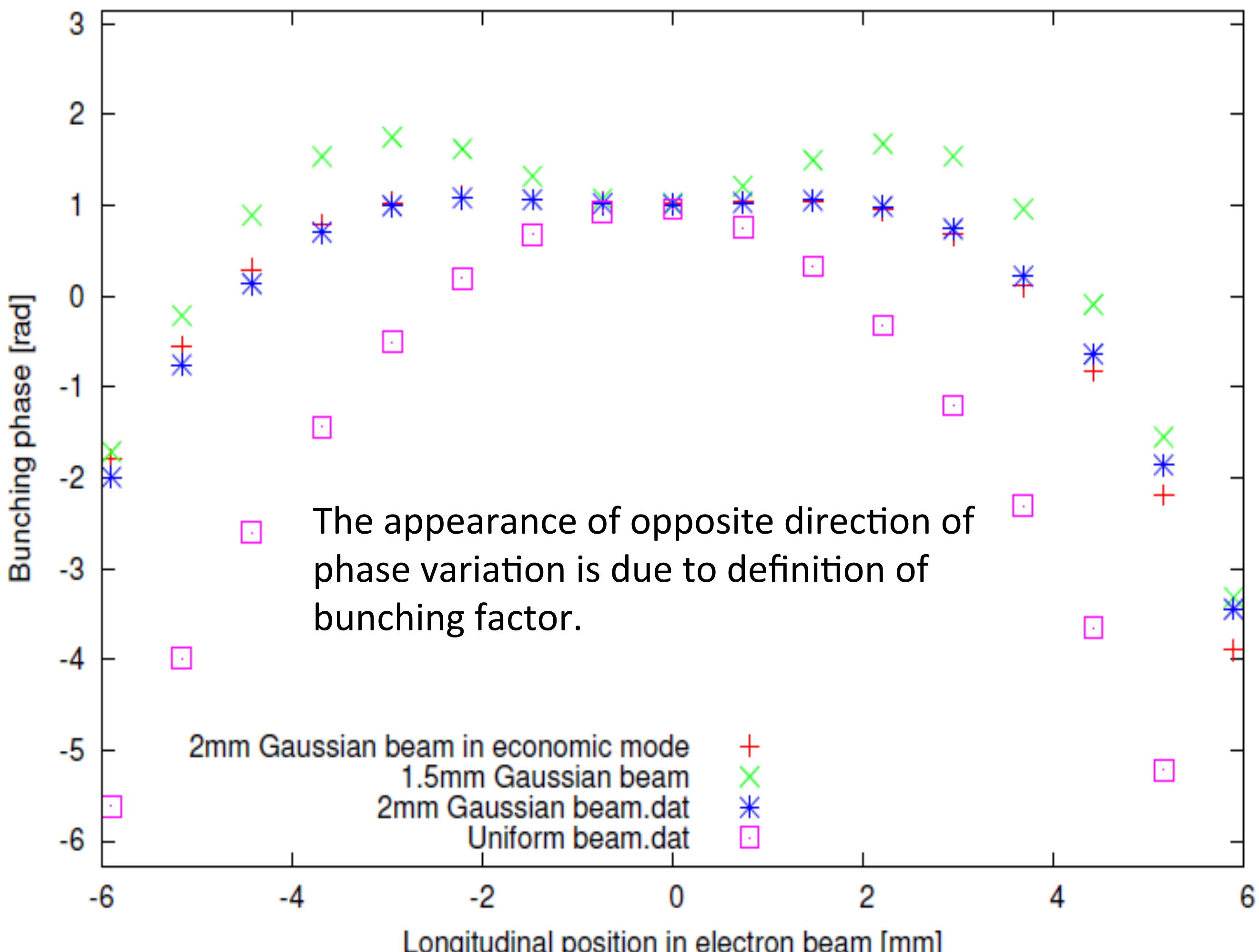


-20 ps

20 ps

Limitation to Amplification: Saturation

- Self-consistent description of e-beam instability includes set of two equations: Maxwell and Vlasov

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial \vec{x}}\cdot\frac{d\vec{x}}{dt}+\frac{\partial f}{\partial \vec{P}}\cdot\frac{d\vec{P}}{dt}=0;$$

$$\frac{\partial F^{ik}}{\partial x^k}=-\frac{4\pi}{c}j^k \quad\Leftrightarrow\left\{\begin{array}{c}\Delta\vec{A}+\dfrac{\partial^2}{c^2\partial^2 t}\vec{A}=-\dfrac{4\pi}{c}\vec{j}\\ \Delta\varphi+\dfrac{\partial^2}{c^2\partial^2 t}\varphi=-\dfrac{4\pi}{c}\rho\end{array}\right\}; \qquad \text{with}\quad j^k=\left(c\rho,\vec{j}\right)\equiv e\int\frac{dx^k}{dt}f\cdot d\vec{P}^3;$$

- It can be linearized when $f=f_o+\varepsilon\cdot\tilde{f},\quad \left|\varepsilon\cdot\tilde{f}\right|/f_o<<1$

$$f=f_o+\varepsilon\cdot\tilde{f};\quad \vec{A}=\vec{A}_o+\varepsilon\cdot\delta\vec{A};\ \varphi=\varphi+\varepsilon\cdot\delta\varphi$$

$$H=H_o+\varepsilon\cdot\tilde{H};\qquad H_o=\sqrt{m^2c^4+\left(c\vec{P}-e\vec{A}_o\right)^2}+e\varphi_o;$$

$$\frac{\partial f_o}{\partial t}+\frac{\partial f_o}{\partial \vec{x}}\cdot\frac{\partial H_o}{\partial \vec{P}}-\frac{\partial f_o}{\partial \vec{P}}\cdot\frac{\partial H_o}{\partial \vec{x}}=0;\quad \tilde{H}=\left(\frac{c\vec{P}-e\vec{A}_o}{H_o-e\varphi_o}\delta\vec{A}+e\delta\varphi\right);$$

Ignored due to the assumption that ε is small

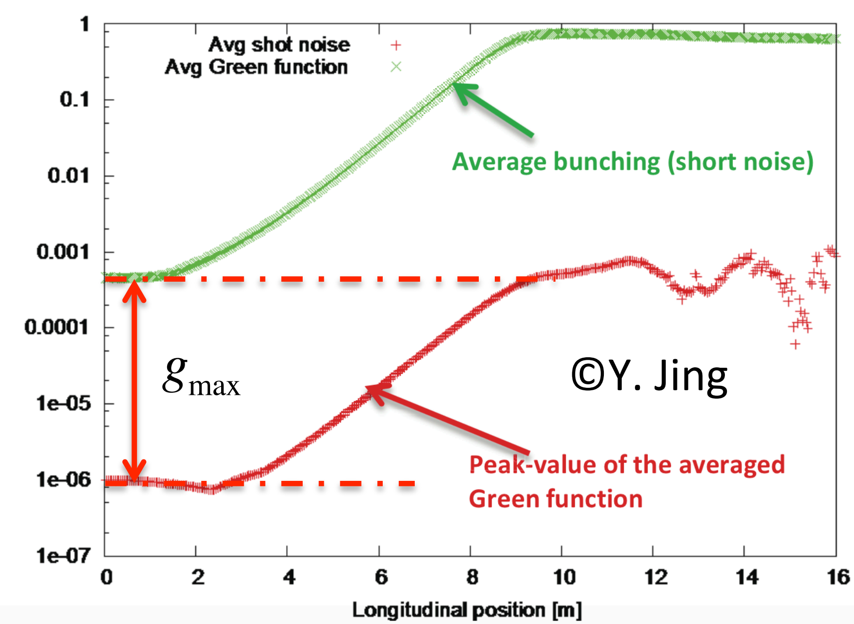
$$\frac{\partial \tilde{f}}{\partial t}+\frac{\partial \tilde{f}}{\partial \vec{x}}\cdot\frac{\partial H_o}{\partial \vec{P}}-\frac{\partial \tilde{f}}{\partial \vec{P}}\cdot\frac{\partial H_o}{\partial \vec{x}}+\frac{\partial f_o}{\partial \vec{x}}\cdot\frac{\partial \tilde{H}}{\partial \vec{P}}-\frac{\partial f_o}{\partial \vec{P}}\cdot\frac{\partial \tilde{H}}{\partial \vec{x}}\cancel{+\varepsilon\left(\frac{\partial \tilde{f}}{\partial \vec{x}}\cdot\frac{\partial \tilde{H}}{\partial \vec{P}}-\frac{\partial \tilde{f}}{\partial \vec{P}}\cdot\frac{\partial \tilde{H}}{\partial \vec{x}}\right)}=0.$$

Onset of Saturation for the Proof of CeC Principle Experiment

$$\left|\delta\hat{n}/n_0\right|_{\max} < 1 \Rightarrow \quad |g|_{\max} < \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}} \Rightarrow \quad g_{\max} \sim 72\cdot\sqrt{\frac{I_e[A]\cdot\lambda_o[\mu m]}{M_c}} = 429$$

$$M_c \equiv \frac{L_c}{\lambda_1} = \frac{1}{\lambda_1 g_{\max}^2}\int_{-\infty}^{\infty}|g(z)|^2\,dz$$

- γ=21.8
- Peak current: 100 A
- Norm emittance 5 mm mrad
- RMS energy spread 1e-3
- λw=4 cm
- a_w = 0.4
- λo=12.7 um
- Mc = 35.8



3D Genesis simulation shows that the maximal gain in bunching factor is 409, which agrees with our estimation.

Model for Kicker: Uniform Electron Beam

- Dynamic equation in Kicker is very similar to that in the modulator except the initial modulation in 6D phase space dominates the process. For κ-2 velocity distribution, the electron density perturbation is determined by:

$$\frac{d^2}{dt^2}\tilde{R}_1\left(\vec{k},t\right)+\omega_p^2\tilde{R}_1\left(\vec{k},t\right)=Z_i\omega_p^2 e^{-\lambda\left(\vec{k},\vec{v}_0\right)t}-\omega_p^2\int_{-\infty}^{\infty}\tilde{f}_1\left(\vec{k},\vec{v},0\right)e^{-\lambda\left(\vec{k},\vec{v}\right)t}d^3v$$

with $\tilde{R}_1\left(\vec{k},t\right)\equiv\tilde{n}_1\left(\vec{k},t\right)e^{-\lambda\left(\vec{k}\right)t}-\int_{-\infty}^{\infty}\tilde{f}_1\left(\vec{k},\vec{v},0\right)e^{-\lambda\left(\vec{k},\vec{v}\right)t}d^3v$ and

$$\lambda\left(\vec{k},\vec{v}\right)\equiv i\vec{k}\cdot\vec{v}-\sqrt{\left(k_x\sigma_x\right)^2+\left(k_y\sigma_y\right)^2+\left(k_z\sigma_z\right)^2}$$

The solution of this inhomogeneous 2nd order differential equation reads

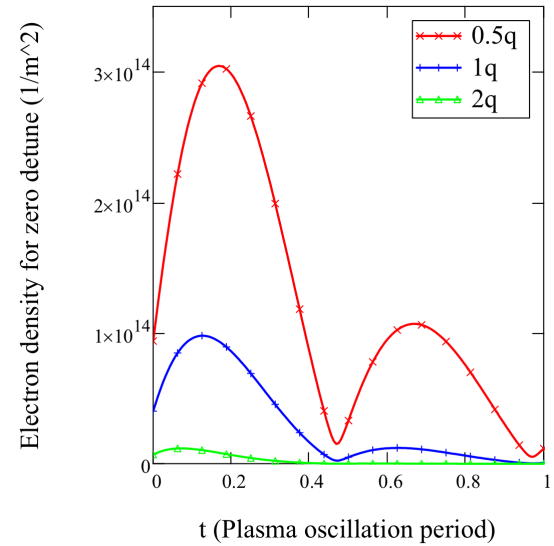
$$\tilde{R}_1\left(\vec{k},t\right)=c_1\cos\left(\omega_p t\right)+c_2\sin\left(\omega_p t\right)$$
$$+\frac{1}{\omega_p}\int_{-\infty}^{\infty}\frac{\omega_p e^{-\lambda t}+\lambda\sin\left(\omega_p t\right)-\omega_p\cos\left(\omega_p t\right)}{\lambda^2+\omega_p^2}\tilde{f}_1\left(\vec{k},\vec{v},0\right)d^3v$$

An example: 1D Initial Modulation

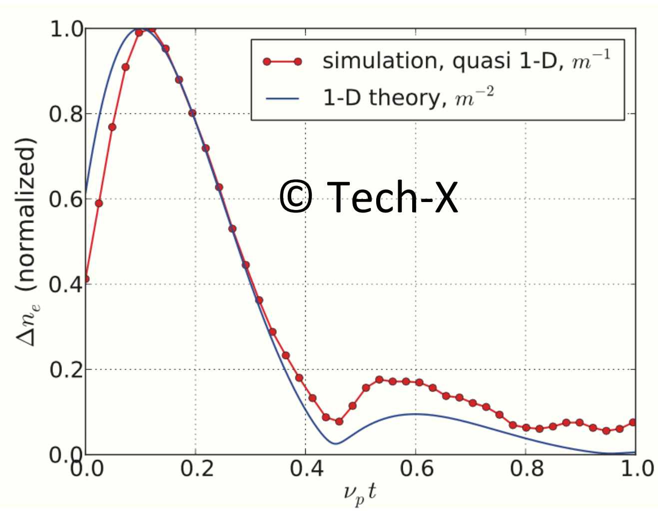
- For 1D FEL output with the following initial transverse perturbation,

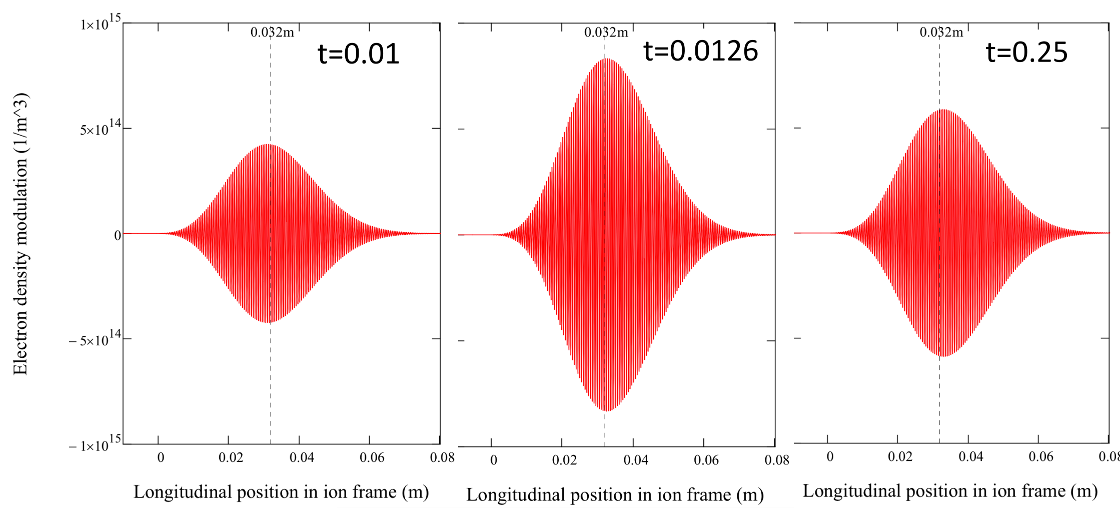
$$\tilde{f}_1(x,y,z,v_x,v_y,v_z,0)=\frac{1}{\pi\beta_x\beta_y}\tilde{f}_1(z,v_z,0)\left(1+\frac{v_x^2}{\beta_x^2}+\frac{v_y^2}{\beta_y^2}\right)^{-\frac{3}{2}}$$

$$\boxed{\tilde{n}_1(k_z,t)=-Z_i\tilde{\Lambda}_{drive}(k_z)e^{ik_z v_{0z}t}e^{-|k_z|\sigma_{vz}t}\left[\cos(\omega_p t)+\frac{(\lambda_1+i\hat{C})\beta_z c\gamma_z\Gamma-ik_z v_{0z}+|k_z|\sigma_{vz}}{\omega_p}\sin(\omega_p t)\right]}$$



- The analytical result has been used to benchmark simulation studies performed by Tech-X.





Field Reduction due to Finite Transverse Modulation Size

- The transverse profile of the modulation can affect the longitudinal field of the wave-packet:

$$\rho(\vec{r}) = \rho_o(r)\cdot\cos(kz);$$

$$\Delta\varphi = 4\pi\rho \Rightarrow \varphi(\vec{r}) = \varphi_o(r)\cdot\cos(kz);$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi_o}{dr}\right) - k^2\varphi_o = 4\pi\rho_o(r)$$

$$\rho(r) = \rho(0)\cdot g(r/\sigma)$$

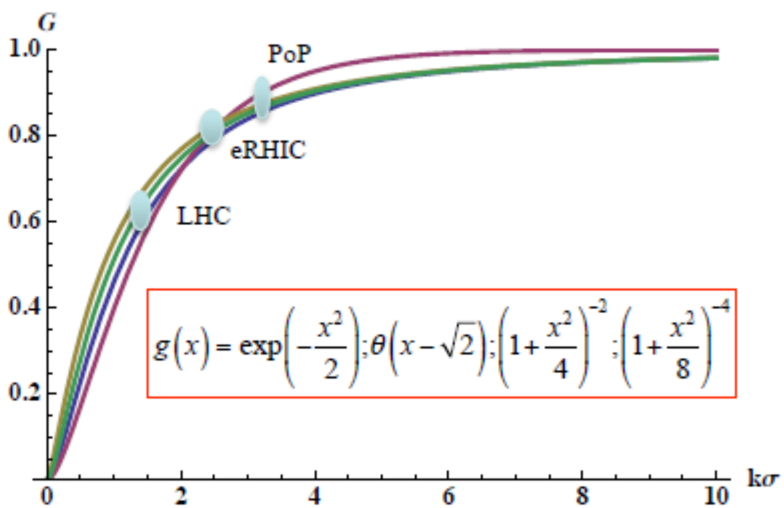
$$E_{zo}(r=0) \propto -\frac{4\pi\tilde{q}}{\sigma^2}G(k_{cm}\sigma)$$

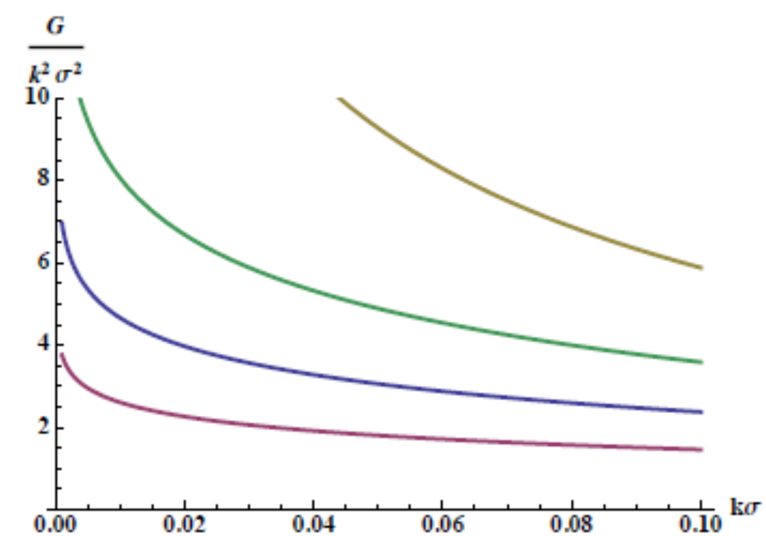
$$\varphi(\vec{r}) = -4\pi\cos(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$E_z = -\frac{\partial\varphi}{\partial z} = -4\pi k\sin(kz)\left\{I_0(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi + K_0(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$E_r = -\frac{\partial\varphi}{\partial r} = 4\pi k\cos(kz)\left\{I_1(kr)\int_r^\infty \xi K_0(k\xi)\cdot\rho_o(\xi)d\xi - K_1(kr)\int_0^r \xi I_0(k\xi)\cdot\rho_o(\xi)d\xi\right\}$$

$$\boxed{k_{cm}\sigma_\perp = \frac{k_o}{\gamma_o}\sqrt{\frac{\beta_\perp\varepsilon_{n\perp}}{\gamma_o}} = \sqrt{\gamma_o}\sqrt{\beta_\perp\varepsilon_{n\perp}}\,\frac{k_w}{2(1+a_w^2)}}$$





- The results reproduces what previously derived by G. Stupakov

Status of analytic studies

What has been worked on...

- Analytical approach to model the three sections of CeC has been explored both for cold and warm electron beam with uniform spatial distribution. The results from the analytical studies have been used to benchmark numerical simulation and to understand the relevant physics process.
- Recently, we explored various effects that could adversely affect the cooling performance, including the long-range longitudnal space charge field in the modulator, saturation of FEL amplification, and field reduction due to finite transverse size of the electron density wave-packet. No show-stoppers have been found.

What is on the to do list ...

- Influences of wake fields on the CeC processes;
- Feasibility studies of adjusting the gain of FEL amplifier via phase shifter;
- Proceeding on modeling CeC for a finite electron beam

...

Numerical Model of the modulator via FFT

For the unitary point charge moving along a straight line $\vec{y}(t) = \vec{x}_0 + \vec{v}_0 t$ we have for the induced electron density perturbation for any number of spacial dimensions d:

$$n_1(\vec{x}, t) = L^{-1}F^{-1}\left[\frac{e^{-i\vec{k}\vec{x}_0}}{\left(\frac{1}{LF_{\vec{k}t}(tf_0(\vec{v}))}\frac{1}{f_d v_{rms}^d}+1\right)\left(s + i\vec{k}\cdot\vec{v}_0\right)}\right] \qquad (1)$$

where $LF_{f_0}\left(\vec{k}t, t\right)$ depends on equilibrium distribution:

$$LF_{\vec{k}t}(tf_0(\vec{v})) = \int_0^\infty e^{-st}\int f_0(\vec{v})\, e^{-i\vec{k}\vec{v}t}\, d\vec{v}\, dt,$$

$\frac{1}{f_d v_{rms}^d}$ is a dimensionless factor, and L^{-1}, F^{-1} are the inverse Laplace and Fourier transforms, respectively.

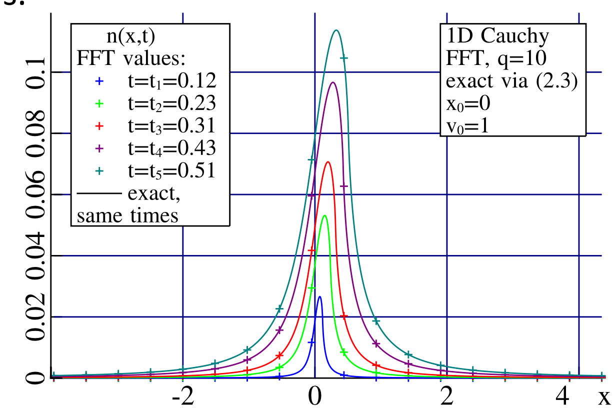
For the 1D Cauchy distribution, it is possible to invert transforms analytically and obtain exact solution:

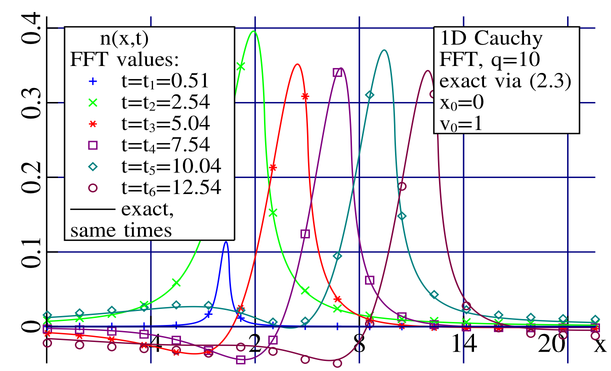
$$\begin{aligned} n_1(\vec{x}, t) = \frac{1}{4\pi}\frac{1}{v_0 - i}\Big(& e^{-\mathcal{A}_+}\left(\mathrm{Ei}(\mathcal{A}_+) - \mathrm{Ei}(\mathcal{B}_+)\right) + \\ & + e^{\mathcal{A}_+}\left(\mathrm{E}_1(\mathcal{A}_+) - \mathrm{E}_1(\mathcal{B}_+)\right)\Big) + \\ + \frac{1}{4\pi}\frac{1}{v_0 + i}\Big(& e^{-\mathcal{A}_-}\left(\mathrm{Ei}(\mathcal{A}_-) - \mathrm{Ei}(\mathcal{B}_-)\right) + \\ & + e^{\mathcal{A}_-}\left(\mathrm{E}_1(\mathcal{A}_-) - \mathrm{E}_1(\mathcal{B}_-)\right)\Big), \end{aligned}$$

$$\text{where } \mathcal{A}_\pm = \frac{tv_0 - x + x_0}{1 \pm iv_0}, \quad \mathcal{B}_\pm = \frac{x_0 - x \pm it}{1 \pm iv_0},$$

and $\mathrm{E}_1(z)$ and $\mathrm{Ei}(z)$ are the exponential integral functions, which can be computed via series expansions.

The Vlasov-Poisson system for the shielding problem in the infinite plasma can be solved via the Laplace and Fourier transforms.





On these Figs we compare exact solution with the numerical one for the 1D Cauchy distribution. Solid lines represent exact solution, the crosses – numerical. Numerical solutions can be obtained for other 1D, 2D and 3D distributions.

Numerical solution of the Vlasov-Poisson system for the finite beam.

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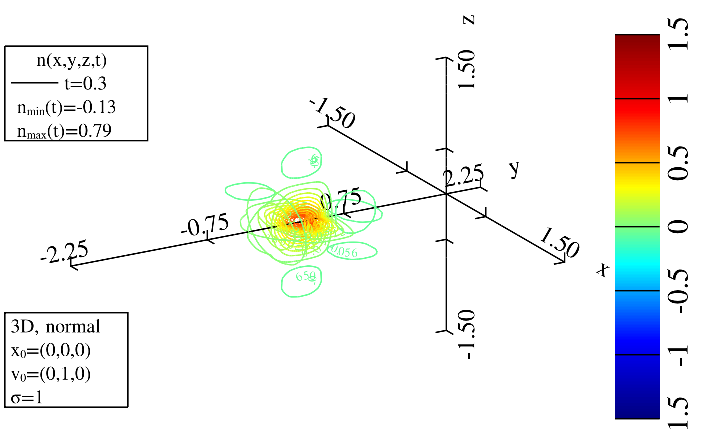
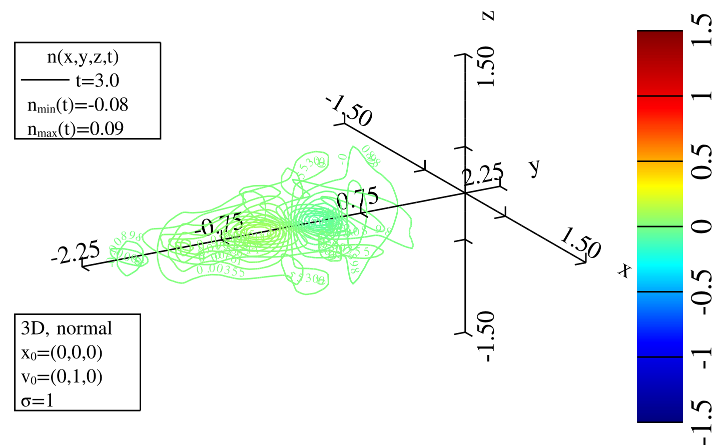
- It's very important to be able to model the modulator for the realistic finite beam.
- We developed a numerical solver for the Vlasov-Poisson system for the finite beam with confining fields. The solver was tested on the exactly solvable equations (non-physical).
- We obtained numerical results for the 1D, 2D and 3D balls with the normal velocity and spatial distributions. Below we present the results for the 3D case:





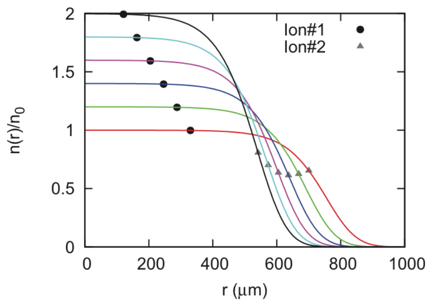
The finite plasma results have some distinctive qualitative features:

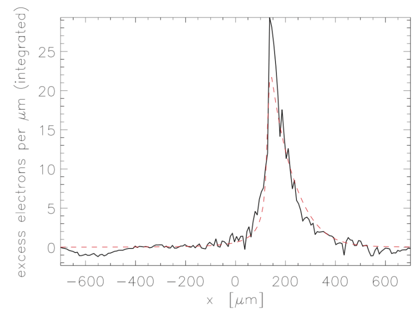
- Due to finiteness and conservation of the total charge, the perturbation is always accompanied by the negative peak.
- The plasma waves are reflected for the plasma's effective boundary.
- There are oscillations of the shape of the perturbations in the confining fields.

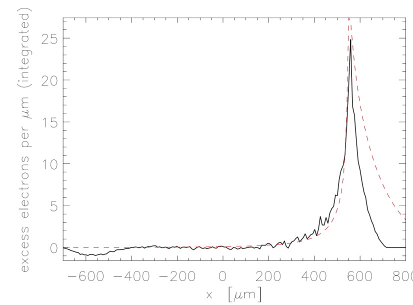
Status of one pass simulation

What has been done...

- Modulator simulation with uniform/continous external focusing







- Genesis simulation of the FEL amplifier with pre-determined bunching factor and energy bunching factor.
- Kicker simulation is based on 3D Vorpal simulation, which agrees with theoretical results at 1-D limit.

What is needed...

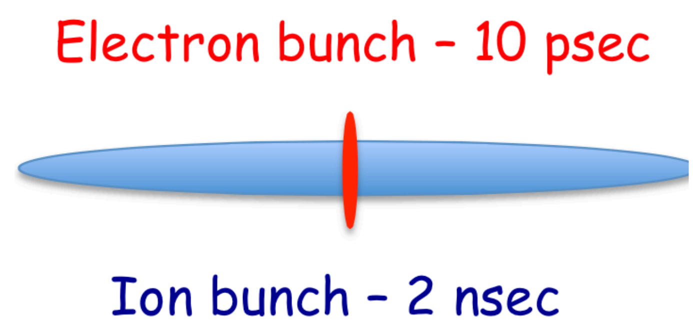
- Modulator simulation with realistic focusing from quadrupoles
- Couple modulator simulation into Genesis simulation, with macro-particles direcly as input (not bunching factors).
- Eventually, we need kicks (as a function of location within the bunch) induced by an ion, which can be directly used in cooling simulation.

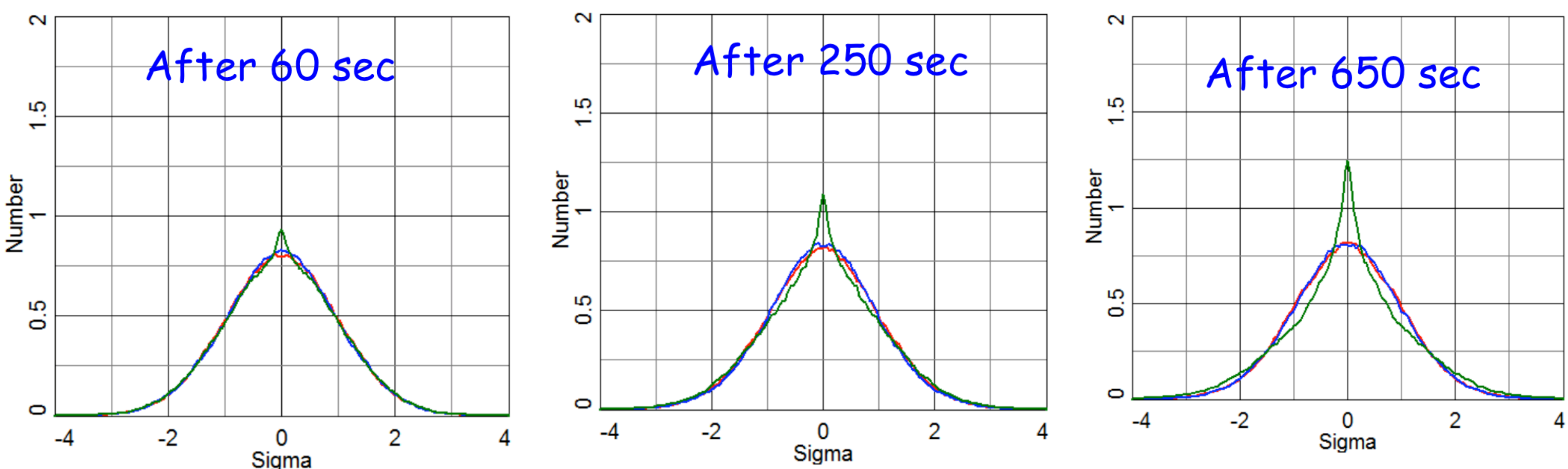
Most of the simulation has been done by Tech-X. However, funds is limited and we presently do not have funds to support this direction at Tech-X.

Simulation of ion beam under cooling

- For the proof of CeC principle experiment, the estimated local cooling time is ~4 seconds.

- Improvement of the cooling simulation requires one turn simulation of the field generated by a single ion as a function of the ion's 6D coordinates, which is work in progress.





Modeling of cooling is performed with betacool by A. Fedotov

Summary

- Analytical studies of CeC proceeds well and we have not found any show-stopper based on these studies.

- Single-pass simulation has been validated by the analytical results for uniform beam. Simulation studies for more realistic electron beam are to be completed.

- Improvement of the cooling simulation is work in progress, which requires input from single-pass simulation.

Backup slides

CeC Parameters

Modulator

Parameter	**CeC PoP**	**eRHIC**	**LHC**
Spices	Au	p	p
Particles per bunch	10^9	$2x10^{11}$	$1.7x10^{11}$
Energy GeV/u	40	250	7,000
RMS ε_n, mm mrad	2.5	0.2	3
RMS energy spread	$3.7\ x10^{-4}$	10^{-4}	10^{-4}
RMS bunch length, nsec	3.5	0.27	1
e-beam energy MeV	21.8	136.2	3812
Peak current	75	50	30
RMS ε_n, mm mrad	5	1	1
RMS energy spread	$1\ 10^{-4}$	$5\ 10^{-5}$	$2\ 10^{-5}$
RMS bunch length, nsec	0.05	0.27	1
Modulator length, m	**3**	**10**	**100**
Plasma phase advance, rad	**1.7**	**2.14**	**0.06**
Buncher	**None**	**None**	**Yes**
Induced charge, e	**88.1**	**1.54**	**2**

CeC Parameters: FEL amplifier

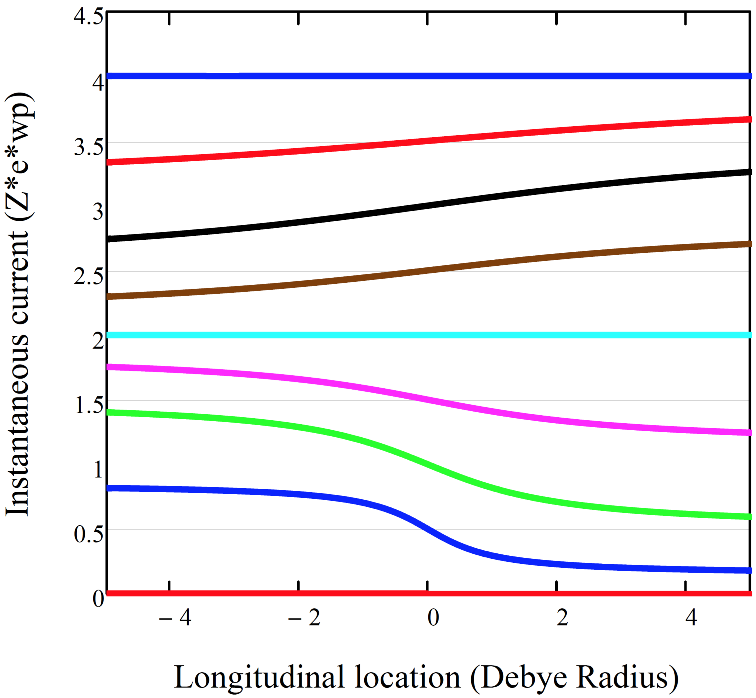
Parameter	CeC PoP	eRHIC	LHC
Spices	Au	p	p
Particles per bunch	10^9	$2x10^{11}$	$1.7x10^{11}$
Energy GeV/u	40	250	7,000
RMS ε_n, mm mrad	2.5	0.2	3
RMS energy spread	$3.7 x10^{-4}$	10^{-4}	10^{-4}
RMS bunch length, nsec	3.5	0.27	1
e-beam energy MeV	21.8	136.2	3812
Peak current	75	50	30
RMS ε_n, mm mrad	5	1	1
RMS energy spread	$1\ 10^{-4}$	$5\ 10^{-5}$	$2\ 10^{-5}$
RMS bunch length, nsec	0.05	0.27	1
λ_w, cm	**4**	**3**	**10**
λ_o, nm	**13,755**	**423**	**91**
a_w	**0.5**	**1**	**10**
g_{max}	**650**	**44**	**17**
$g_{required}$	**100**	**3**	**8.5**
FEL length, m	**7.5**	**9**	**100**
Bandwidth, Hz	**$6.2\ 10^{11}$**	**$1.1\ 10^{13}$**	**$2.4\ 10^{13}$**

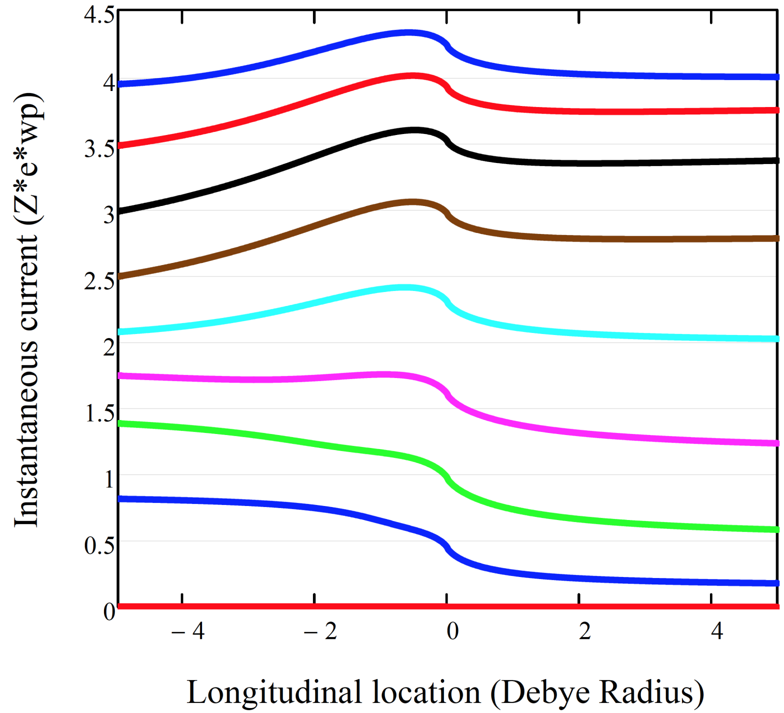
Infinitely Wide Beam

$$\dot{I}_d(z,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} dx\,dy e^{i\vec{k}\cdot\vec{x}} \dot{\tilde{j}}_{d,z}(\vec{k},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_z \cdot e^{ik_z z} \dot{\tilde{j}}_{d,z}(0,0,k_z,t)$$

$$\sum_i \frac{\delta E_i}{E_0} = \sum_i \frac{v_{i,z}}{c} = -I_{d,z}(z,t)\cdot\frac{\Delta z_{slice}}{ec} \qquad I_d(z,t) = \frac{Z_i e\omega_p}{\pi}\left[\sin(\omega_p t)\arctan\left(\frac{v_{0,z}}{\bar{\beta}} + \frac{z}{\bar{\beta}t}\right) - \int_0^t d\tau \frac{v_{0,z}\bar{\beta}\tau\sin(\omega_p\tau)}{\bar{\beta}^2\tau^2 + (z+v_{0,z}\tau)^2}\right]$$

Two adjacent curves are 1/8 plasma oscillation apart, i.e. $\omega_p t = 0,\ \pi/4,\ \pi/2\ldots$



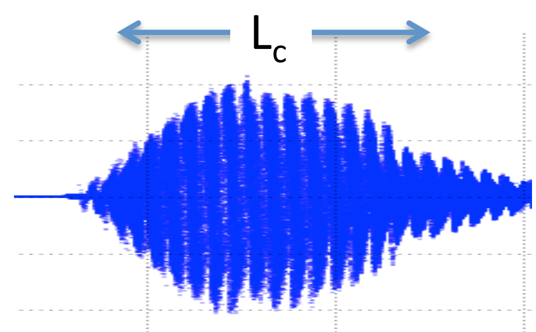


Validity of Linearization in FEL

- In the linear region of a FEL, the response of the FEL to a delta-like density perturbation can be expressed as:

$$\delta(z - z_{i0}) \rightarrow \delta(z - \overline{z}_i) + G(z, z_{i0}, t)$$

and the overall response to the beam shot noise is obtained by applying the superposition principle.



$$\left\langle \delta\tilde{n}(z,t)^2 \right\rangle < \left\langle \tilde{n}_0(z,t) \right\rangle^2$$

$$\overline{n}_0(z,t)^2 > \int_{-\infty}^{\infty} \left[G(z, z_0, t) \right]^2 \overline{n}(z_0) dz_0 + \left[\int_{-\infty}^{\infty} G(z, z_0, t) \overline{n}(z_0) dz_0 \right]^2$$

$$|G|_{\max} < \sqrt{\frac{\overline{n}_0}{L_c}} = \sqrt{\frac{I_e}{ecL_c}}$$

We define the coherent length as:

$$L_c \equiv \frac{1}{G_{\max}^2} \int_{-\infty}^{\infty} G(\xi)^2 \, d\xi$$

which is the characteristic width of the Green function.

Limitation on bunching factor

- For an narrow band amplifier such as FEL, the Green function and the line density variation in the linear region have the following form:

$$\bar{n}(z,t) = n_0 + \delta\hat{n}(z,t)\cos\left(k_o z + \psi(z,t)\right) \qquad G(z,t) = \hat{G}(z,t)\cos\left(k_o z + \phi(z,t)\right)$$

- Hence, the bunching factor is often used to describe the line density variation.

$$N_\lambda \left|b(z_0)\right| = \left| \int_{z_0 - \frac{\pi}{k_o}}^{z_0 + \frac{\pi}{k_o}} \bar{n}(z) e^{ik_o z}\, dz \right| \approx \frac{\lambda_o}{2}\left|\delta\hat{n}(z_0)\right| \qquad k_o = \frac{2\pi}{\lambda_o} \qquad \lambda_o = \frac{\lambda_w\left(1 + a_w^2\right)}{2\gamma_e^2}$$

- We can define a similar quantity to describe the amplitude of the Green function:

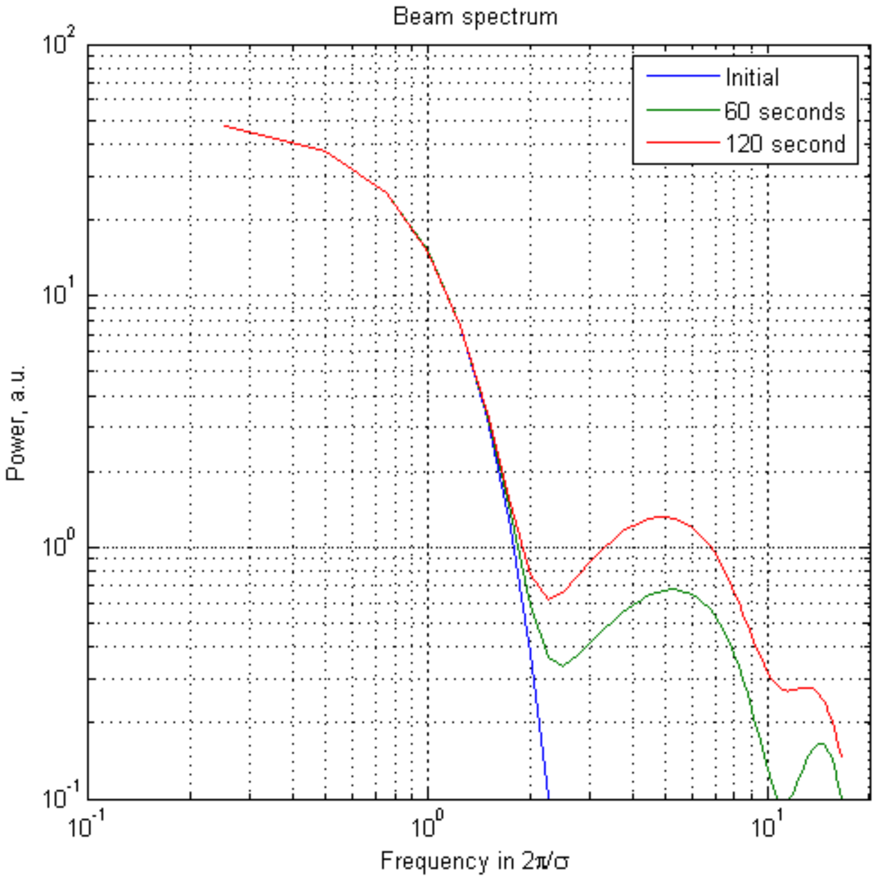
$$\left|g(z_0)\right| = \left| \int_{z_0 - \frac{\pi}{k_o}}^{z_0 + \frac{\pi}{k_o}} G(z) e^{ik_o z}\, dz \right| \approx \frac{\lambda_o}{2}\left|\hat{G}(z_0)\right|$$

- The condition for the newly defined variables become

$$\left\langle \delta\tilde{n}(z)^2 \right\rangle < \left\langle \tilde{n}_0(z) \right\rangle^2 \Rightarrow \left|b(z_0)\right| < \frac{1}{2} \qquad \left|G(z_0)\right| < \sqrt{\frac{I_e}{ecL_c}} \Rightarrow |g|_{\max} < \frac{\lambda_o}{2}\sqrt{\frac{I_e}{ecL_c}}$$

Simulation of cooling an ion bunch (Betacool)

- For the Au ions r.m.s. bunch length is 1.5 ns
- That makes r.m.s. length of the cooled part 80-120 ps
- The cooling effects can be observed with oscilloscope 2 GHz or more bandwidth or spectrum analyzer with similar upper frequency



Courtesy I. Pinayev and A. Fedotov